CS 70 Worksheet: Graphs

1 Trees

One of the most important facts about trees is that a tree with n vertices has exactly $n-1$ edges. Let's try and prove this, and practice induction along the way.

1. What can we induct on? What's the base case in this situation?

2. Let's prove the inductive step. Start out with your hypothesis for k and then prove it true for $k+1$. Avoid build-up error by starting with an object that the $k+1$ case applies to, then somehow find an object inside it that the k case applies to. (Hint: what happens if you remove a vertex from the tree with $(k+1)$ vertices? What can you say about the remainder of the tree?)

2 Colorings

Colorings are an important part of graphs, so let's try to prove some things about them.

1. A graph is *bipartite* if its vertices can be divided into two sets, where the only edges in the graph "cross" the two sets (there exists no edge between vertices that both live in one of the set). Show that a graph is bipartite if and only if it is 2-colorable.

2. Show that the hypercube of size 2^n can be 2-colored (Hint: prove with induction, think about the recursive structure of the hypercube!)

3. Consider the node of a tree with maximum degree, call this the root node and let it have degree k. Show that the tree is k-colorable.

3 True/False

It's common on CS 70 midterms to see lots of True/False questions. I pulled some of these from past midterms. First, see what your gut (some people call this intution) tells you, *write that down*, then think about it and see if your answer changed. Also try to prove the question for yourself. We'll see how useful your gut is at the end!

1. Any graph where $|E| \leq 3|V| - 6$ is planar.

2. If a graph is k-colorable, adding any edge to it makes it $(k+1)$-colorable at worst.

3. If all vertices of a graph have degree exactly four, then the graph must be K_5.

4. Any graph with a vertex of degree d can be d + 1 colored.

5. A graph with k edges and n vertices has a vertex of degree at least $2k/n$.

6. Consider graphs that satisfy a "triangle property": if vertices u and v are connected, and v and w are connected, then u and w are connected. If a graph satisfies this property, it must be fully connected.

7. Every graph has an even number of odd-degree vertices.

8. The maximum degree of a vertex in a planar graph is 6.

9. All complete graphs K_n where n is even have Eulerian tours.

4 Proof Problems

Some more graph problems that ask you for proofs.

1. Consider a fully connected directed graph where between every pair of vertices u, v, either u is connected to v or v is connected to u. Prove that every vertex must be connected directly to the vertex with maximum in-degree with a path of length at most 2.

2. Consider a graph with n vertices that has a path of the form $v_1 \dots v_n$ involving all the vertices, and where v_1 and v_2 both have degree at least $n/2$. Show that the graph has a hamiltonian cycle (a cycle that visits every node in the graph at least once).

3. Show that every finite connected directed graph where every vertex has in-degree at least one contains a directed cycle.

4. Show that the shortest path between two points in a hypercube has length n, always.

5. Consider n undirected graphs $G_1, G_2, ...G_n$ that share no vertices or edges and have exactly two odd- degree vertices each. Prove that it is possible to construct an Eulerian tour visiting all of $G_1, G_2, ...G_n$ using only n additional edges to connect them.