CS 70: Un__tability, Counting, Probability

1 Uncountability

1. Is the powerset of $\mathbb{N}$ countable (the set of all subsets of $\mathbb{N}$)? How would you prove this?

   Solution:

   No, do a proof by diagonalization. Assume that the powerset is countable, then we must have an infinite listing of subsets of $\mathbb{N}$: $S_0, S_1, \ldots$.

   Define a subset of the naturals S' by letting $x \in S'$ iff $x \notin S_x$. This is basically inverting the diagonal. Then this subset has some index i in the listing, but $i \in S'$ iff $i \notin S_i$ iff $i \notin S'$, which is always a contradiction.

2. Are the integers $\mathbb{Z}$ countable? How about pairs of integers (x, y) where $x = 0$ or $y = 0$?

   Solution:

   Yes, see the proof presented in lecture. This constructs some bijective function that maps the even integers to positive integers and odd integers to negative ones.

   There is some similar proof for the pairs with one coordinate zero, except now working mod 4.

3. Is a countable union of countable subsets countable? This means $\bigcup_i U_i$ where $i \in \mathbb{N}$.

   Solution:

   Yes. A countable union "looks like" $\mathbb{N} \times \mathbb{N}$, let (x, y) map to the yth element of U_x, then this is a valid way to count countable unions.

4. Is the set of all irrational numbers countable?

   Solution:

   No. If it was, then the reals would be a union of two countable sets (the rationals and irrationals) and be countable. This is a contradiction, because as proved in class, the reals aren't countable.

5. Is the set of all programs countable?

   Solution:

   Yes. There are many ways to encode programs with numbers, one of them is ASCII, like giving each letter in the program a bit. But the important thing is we can give at least one number for each possible program.

6. Show that there are numbers in $\mathbb{R}$ that cannot be computed. (Wow!!!)

   Solution:

   We showed the set of programs is countable. The set of inputs is countable too, so the set of all pairs of programs and inputs we could run them on (P, x) is also countable. Then, since each pair (P, x) corresponds to one possible output the program could give, the set of all possible outputs all possible programs could give on all possible inputs is still countable! But inside the reals which are strictly bigger than countable sets, this means we have real numbers that could not be computed by any program! Intuitively, these have to be 'infinite' objects with an infinite amount of information that a program would never give to us in finite time, like π.

2 Uncomputability

1. Consider the following program:

```
def is_mod_2(P):
    if (P implements the mod 2 function):
        return True
    else:
        return False
```

   Show it cannot exist as a program.

   Solution:

   Assume that it does, we can then solve the halting problem with a program! We want to build halt so that it uses `is_mod_2`.

```
def halt(P, x):
    def Q(y):
        P(x)
        return y % 2
    return is_mod_2(Q)
```

We basically modify the program P into Q so that is_mod can tell us whether P halts! Notice that if P halts, then running $P(x)$ will finish, so then Q implements the mod 2 function. But if P doesn't halt, then Q doesn't halt, so it can't implement the is_mod_2.

2. Consider this program:

```
def returns_42_on_42(P):
    x = P(42)
    if x = 42:
        return True
    else:
        return False
```

Can this exist? What if we replace the if condition with `if P(42) eventually halts and gives us 42`?

Solution:

Yes it can! We just wrote it. :) More intuitively, this program calls P, meaning that it doesn't really do any "magic" to analyze P and figure out if it halts or not. This is very different from the program that returns true if $P(42)$ eventually halts, because that one performs the magic to figure out that P does eventually halt, and even if P does not halt, it still will.

3 Counting

1. How many permutations of SUPERMAN are there?

Solution:

There are no repeated letters here, so we just have 8!.

2. How many for ARKANSAS?

Solution:

We can start with 8!, but this makes all the As and Ss mean different objects we are arranging, and we want to consider them as the same. We can divide by 3! to get rid of overcounting the As and divide by 2! to get rid of the Ss.

3. We have 5 cookies we are trying to divide between 3 students. How many ways are there to divide the cookies among all the students?

Solution:

This becomes a stars and bars problem. There are 2 bars, not 3 (because 2 bars makes 3 'compartments' to give people cookies), and 5 stars for each cookie. Then this is $\binom{7}{2}$, as we are just choosing the locations for the 2 bars which determines how many cookies each person gets.

4. How many ways are there if we want to give every student at least one cookie?

Solution:

First give each student a cookie! Now we are just trying to distribute 2 cookies among 3 students, ignoring the 3 cookies we just gave, which is $\binom{4}{2}$.

5. Let p, q be prime. How many numbers are there among $1, 2, \ldots (pq)^2$ that are relatively prime to pq?

Solution:

This is an example of inclusion, exclusion. $p, 2p, \ldots, pqpq$ are not relatively prime to p, and $q, 2q, \ldots, pqpq$ are not relatively prime to q. That gives pq^2 numbers not relatively prime to p, and qp^2 not relatively prime to q. However, we have double counted the numbers that both p and q divide! These are $pq, 2pq, \ldots pqpq$, of which there are pq. So in total we have $pq^2 + qp^2 - pq$ numbers not relatively prime to p or q. But to find the numbers that are, we subtract that from all the numbers, $pqpq$, so we end up with $pqpq + pq - pq^2 - qp^2$.

6. How many combinations of even natural numbers (x_1, x_2, x_3, x_4) are there such that $x_1 + x_2 + x_3 + x_4 = 20$?

   Solution:

   This is a stars and bars in disguise! Notice that if all the numbers are even, for example $4, 4, 4, 6$, we can divide both sides of the equation to obtain $2 + 2 + 2 + 3 = 10$. So this is just counting how many sets of 4 non-negative natural numbers add up to 10, which is stars and bars. There are 10 stars and 4 compartments / 3 bars, so we have $\binom{13}{3}$.

7. There is a class with 2n children where n are boys and n are girls. How many ways are there to arrange them in a line so that they alternate by gender?

   Solution:

   We already know that the boys and girls alternate by gender, i.e. BGBGBG or GBGBGB. Then either of these sequences are completely determined by what order we put the boys in and what order we put the girls in, which is $(n!)^2$. That is only counting one of the possibilities though, i.e. we just counted how many ways we could have BGBGBG. Multiply by 2, so $2(n!)^2$ to get the final answer.

8. How many ways are there to arrange them where all the girls are before all the boys?

   Solution:

   This is a similar case, the sequence is completely determined by how we arrange the girls independently from arranging the boys, so $(n!)^2$.

9. How many ways are there to arrange them so that all the girls are in an uninterrupted block? (there is no boy in between two girls)

   Solution:

   This is kind of like arranging $n + 1$ things, where n of them are boys, and one is the giant block of n girls. There are $n + 1$ ways to do this. Then it looks like the previous problems, now it just depends how we order the girls and boys within their respective genders, so we end up with $(n + 1) \cdot n! \cdot n!$.

10. How many ways are there for neither the girls nor the boys to stand in an uninterrupted block?

    Solution:

    This is inclusion/exclusion. There are $(2n)!$ ways to arrange the girls and boys in general. So let's subtract from that the ways that either the girls or the boys stand in an uninterrupted block. Both of these are $(n + 1) \cdot n! \cdot n!$. But we're double counting if the all the boys and all the girls are both uninterrupted, which is number 8, done twice for all the girls before all the boys and all the boys before all the girls, which is $2(n!)^2$. Then the number of ways either the girls or boys stand in an uninterrupted block is $2(n + 1) \cdot (n!)^2 - 2 \cdot (n!)^2 = 2n \cdot (n!)^2$. Finally, we have to subtract that from $(2n)!$ to get $(2n)! - 2n \cdot (n!)^2$.

11. Use a combinatorial argument to prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

    Solution:

    $\binom{n}{k} = \binom{n}{n-k}$. So now let's try to show

    $$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

    The right hand side is just picking n items from $2n$. For the left hand side, split the $2n$ set into two n-sized sets, call them the left and right sets. Then our strategy is to still pick n items, where these items could come from either set. We first have to pick how many come from the left set, call that k. Then the number that come from the right set is $n - k$, so the number of ways to pick n elements where k are from the left and $n - k$ are from the right is $\binom{n}{k}\binom{n}{n-k}$.

12. Give a combinatorial proof of $\binom{k+n-1}{n-1} = \sum_{i=0}^{k} \binom{k-i+n-2}{n-2}$.

    Solution:

    I'll omit the solution because I'll put this on the next worksheet :)