CS 70: Un__tability, Counting, Probability

1 Uncountability

1. Is the powerset of $\mathbb{N}$ countable (the set of all subsets of $\mathbb{N}$)? How would you prove this?

2. Are the integers $\mathbb{Z}$ countable? How about pairs of integers (x, y) where $x = 0$ or $y = 0$?

3. Is a countable union of countable subsets countable? This means $\bigcup_i U_i$ where $i \in \mathbb{N}$.

4. Is the set of all irrational numbers countable?

5. Is the set of all programs countable?

6. Show that there are numbers in $\mathbb{R}$ that cannot be computed. (Wow!!!)

2 Uncomputability

1. Consider the following program:

```
def is_mod_2(P):
    if (P implements the mod 2 function):
        return True
    else:
        return False
```

   Show it cannot exist as a program.

2. Consider this program:

```
def returns_42_on_42(P):
    x = P(42)
    if x = 42:
        return True
    else:
        return False
```

   Can this exist? What if we replace the if condition with `if P(42) eventually halts and gives us 42`?

3 Counting

1. How many permutations of SUPERMAN are there?

2. How many for ARKANSAS?

3. We have 5 cookies we are trying to divide between 3 students. How many ways are there to divide the cookies among all the students?

4. How many ways are there if we want to give every student at least one cookie?

5. Let p, q be prime. How many numbers are there among $1, 2, \ldots (pq)^2$ that are relatively prime to pq?

6. How many combinations of even natural numbers (x_1, x_2, x_3, x_4) are there such that $x_1 + x_2 + x_3 + x_4 = 20$?

7. There is a class with 2n children where n are boys and n are girls. How many ways are there to arrange them in a line so that they alternate by gender?

8. How many ways are there to arrange them where all the girls are before all the boys?

9. How many ways are there to arrange them so that all the girls are in an uninterrupted block? (there is no boy in between two girls)

10. How many ways are there for neither the girls nor the boys to stand in an uninterrupted block?

11. Use a combinatorial argument to prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

12. Give a combinatorial proof of $\binom{k+n-1}{n-1} = \sum_{i=0}^{k} \binom{k-i+n-2}{n-2}$.