Continuous Probability, Joint Distributions

1 Intro

1. How do continuous variables differ from discrete variables?

   Solution: Both take values in $\mathbb{R}$. However, continuous variables have 'enough outcomes' in the probability space to be uncountable.

   Discrete variables are defined in terms of a probability function $P(X = k)$

2. How do we take expectation and variance for continuous variables?

   Solution:
   $$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$
   $$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

   In general for any function of a random variable $\varphi(x)$:
   $$E[\varphi(X)] = \int_{-\infty}^{\infty} \varphi(x) f_X(x) dx$$

3. What are the analogs of the following distributions for continuous random variables?

   (a) Uniform distribution

   Solution: Still the uniform distribution, but just defined on a continuous probability space.

   (b) Geometric distribution

   Solution: The exponential distribution. It has the memoryless property like the geometric distribution.

   (c) Binomial distribution

   Solution: The normal/Gaussian distribution – sort of. It has an easy-to-find mean and variance, and maintains a nice bell curve shape. But it can be negative as well.

4. What are the properties of the CDF? Of the PDF? How do we get one from the other?

   Solution: CDF: $P(X \leq x) = F(x) = \int_{-\infty}^{x} f_X(x) dx$

   PDF: $f_X(x) = F'(x)$.

   The CDF must be between 0 and 1, the limit to the left must be 0, and the limit to the right must be 1. The PDF must be ≥ 0, and $\int_{-\infty}^{\infty} f_X(x) dx = 1$.

5. Show how we can write every normal distribution in terms of the standard normal $N(\mu = 0, \sigma^2 = 1)$.

   Solution: Let the distribution be $X \sim N(\mu, \sigma^2)$. Then $E[X] = \mu$ and $var[X] = \sigma^2$. Consider
   $$\sigma N(0, 1) + \mu$$
   . Then this has the same expectation and variance as X and is normally distributed, so then $X = \sigma N(0, 1) + \mu$, and we get
   $$N(0, 1) = \frac{X - \mu}{\sigma}$$

2 Problems

1. Let X be a random variable with pdf given by

$$f_X(x) = \begin{cases} cx^2 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

.

   (a) Find c that makes this a valid random variable.

   Solution: We want $\int_{-\infty}^{\infty} f_X(x)dx = 1$. So $\int_{-1}^{1} cx^2 = c\left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}c = 1$. Then $c = \frac{3}{2}$.

   (b) Find $E[X]$ and $var[X]$.

   Solution:

$$E[X] = \int_{-1}^{1} x \cdot \frac{3}{2}x^2 dx = \frac{3}{2}\left[\frac{1}{4}x^4\right]_{-1}^{1} = 0$$

   This turns out to be 0! This is because $(-x)^3 = -x^3$, also known as saying x^3 is an *odd* function. To find $var[X] = E[X^2] - E[X]^2$, we first find $E[X^2]$:

$$E[X^2] = \int_{-1}^{1} x^2 \cdot \frac{3}{2}x^2 dx = \frac{3}{2}\left[\frac{1}{5}x^5\right]_{-1}^{1} = \frac{3}{5}$$

   So our variance is just $\frac{3}{5}$.

   (c) Find $P(X \leq \frac{1}{2})$.

   Solution: We need to find $\int_{-\infty}^{\frac{1}{2}} f_X(x)dx$:

$$P(X \leq \frac{1}{2}) = \int_{-1}^{\frac{1}{2}} \frac{3}{2}x^2 dx = \frac{3}{2}\left[\frac{1}{3}x^3\right]_{-1}^{\frac{1}{2}} = \frac{9}{16}$$

2. Let X be a random variable with pdf

$$f_X(x) = \begin{cases} 4x^3 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find $P(X \leq \frac{2}{3} \mid X > \frac{1}{3})$.

   Solution: This is finding a conditional probability. We know:

$$P(X \leq \frac{2}{3} \mid X > \frac{1}{3}) = \frac{P(X \leq \frac{2}{3} \cap X > \frac{1}{3})}{P(X > \frac{1}{3})} = \frac{P(\frac{1}{3} < X < \frac{2}{3})}{P(X > \frac{1}{3})}$$

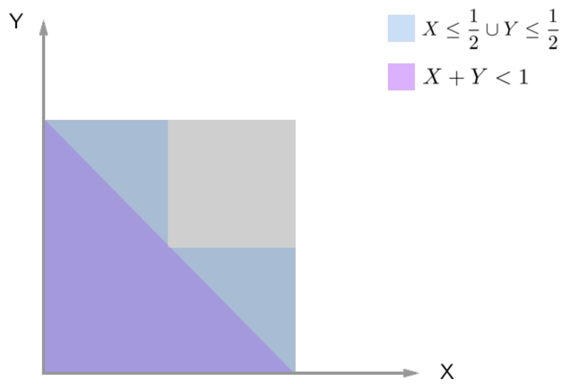
   The rest is evaluating integrals, the top is $\int_{\frac{1}{3}}^{\frac{2}{3}}$ and the bottom is $\int_{\frac{1}{3}}^{1}$.

3. Two real numbers are chosen uniformly from $[0, 1]$. What is the probability that their sum is less than or equal to 1 given that one of them is less than or equal to $1/2$?

   Solution: It's really best to see the solution in a picture. We want to find:

$$P(X + Y \leq 1 \mid X \leq \frac{1}{2} \cup Y \leq \frac{1}{2})$$

   We can summarize this with a diagram like this one:



Then this comes down to comparing areas. The area of where $X+Y<1$ is $\frac{1}{2}$, while for $X \leq \frac{1}{2} \cup Y \leq \frac{1}{2}$

4. Let X and Y be jointly continuous r.vs. with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

. Are X and Y independent? Find $P(X > Y)$.

Solution: Yes, they are. We can write

$$f_{X,Y}(x,y) = 6e^{-(2x+3y)} = 6e^{-2x} 6e^{-2y} = f_X(x) f_Y(y)$$

because we were able to factor the joint distribution into a product of the distributions of X and Y, this means they are independent.

Then finding $P(X > Y)$ becomes finding the integral of $f_{X,Y}(x,y)$ over all points where $X > Y$. This is basically splitting the first quadrant of the plane (where $x, y \geq 0$) into two halves along the diagonal, and we are interested in finding the integral over the bottom one.

$$\int_0^\infty \int_0^x 6e^{-2x+3y} dy dx$$

CS 70 doesn't expect you to evaluate integrals like this one. Sorry if I scared anybody. Almost every time, questions on continuous joint distributions are just about finding the area of shapes we know the formulas for, like problem 3.

5. Let X be a positive continuous r.v. Show that $E[X] = \int_0^\infty P(X \geq x) dx$.

Solution: I realized that this problem does not really help us that much. If you are interested, you can read more here: `https://math.stackexchange.com/questions/63756/tail-sum-for-expectation`